# Noninteger-dimensional hyper-Euclidean lattices on Sierpinski carpets



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COMMENT

# Non-integer-dimensional hyper-Euclidean lattices on Sierpinski carpets

Yoshi-hiro Taguchi

Department of Physics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo 152, Japan

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**Abstract.** Random walks on Sierpinski carpets are studied with bond moving renormalisation. Using the results, we construct fractal lattices whose fractal dimension of random walk is two. Such fractal lattices are called hyper-Euclidean lattices. It turns out that hyper-Euclidean lattices have a connectivity which is one smaller than the fractal dimension.

In recent years, the idea of fractals has been providing several novel concepts in physics (Mandelbrot 1982). Actual construction of non-integer-dimensional hypercubics is one of these new concepts. Gefen *et al* (1983) have proposed such a construction using special examples of translationally invariant Sierpinski carpets. Recently, Hao and Yang (1987) have investigated critical phenomena on a new class of Sierpinski carpets which does not belong to those of Gefen *et al* (1983). In this comment we study the dynamic properties of the carpets proposed by Hao and Yang (1987) using bond moving renormalisation (Gefen *et al* 1983). This renormalisation gives us the fractal dimension of random walk, $d_w$. The results will tell us what kinds of carpets have $d_w$ equal to two. Lattices whose fractal dimension is the same as spatial dimension are called Euclidean lattices. In general, Euclidean lattices ($d$-dimensional cubics and quasicrystals) have $d_w$ equal to two. Hence we call the fractal lattices whose $d_w$ is two hyper-Euclidean (HE) lattices. Construction of HE lattices with Sierpinski carpets is the main purpose of this comment. We will also compare the HE lattices with the hypercubics of Gefen *et al* (1983).

First, we consider the resistor networks on Sierpinski carpets (Gefen *et al* 1983, 1984). As pointed out by Watanabe (1985), the first expression for the resistor networks by Gefen *et al* (1984) is incorrect. Moreover, their second expression (Gefen *et al* 1983) contains a misprint†, and is not enough to deal with the carpets proposed by Hao and Yang (1987). We will therefore make up the renormalisation equations from the outset.

Figure 1 shows an example of the Sierpinski carpets treated in this comment. These carpets consist of only two types of columns. One of these types is the 'complete' one, which contains no eliminated squares. Another is the incomplete one, in which there are $m$ internal bonds and $n$ bordered bonds on the boundary of the eliminated areas;

† There is a misprint in the second term of the first relation in equation (5). $(c-1)$ should be replaced by $(c-l)$.

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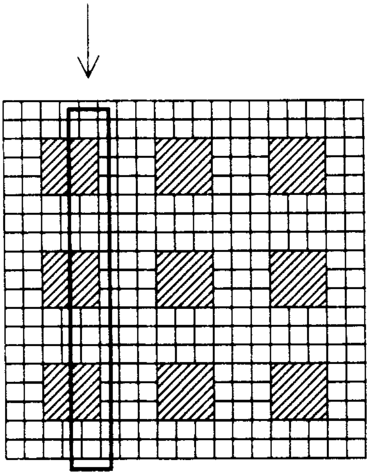

**Figure 1.** An example of Sierpinski carpets, with $b = 19$, $l = 9$, $m = 7$, $n = 6$, fractal dimension $d_f = 1.913$ and connectivity $Q = 0.782$. The arrow represents one of incomplete columns. The shaded squares are eliminated.

the former have resistance $R$, and the latter resistance $R_w$. The resistor-moving renormalisation group scheme (Gefen *et al* 1984) yields the following equations:

$$R' = (b-l)\frac{R}{b} + l\left(\frac{m}{R} + \frac{n}{R_w}\right)^{-1} \tag{1a}$$

$$R'_w = (b-l)\left(\frac{b-1}{2R} + \frac{1}{R_w}\right)^{-1} + l\left(\frac{m-1}{2R} + \frac{n+2}{2R_w}\right)^{-1} \tag{1b}$$

where $b$ is the number of subsquares along a side and the number of eliminated subsquares is $l^2$ (see figure 1). We can rewrite equations (1) as a single recursion relation for the variable $a = R_w/R$. Moreover, this recursion relation has only one positive fixed point; $a^* = 2$†. The location of this fixed point does not depend on parameters $b$, $l$, $m$, $n$. At the fixed point, we can obtain the recursion relation

$$R' = \left(\frac{b-l}{b} + \frac{l}{b-l}\right) R \tag{2}$$

using $2m + n = 2(b-l)$.

Equation (2) gives us the exponent $\zeta$, which is defined as $R \sim b^{\zeta}$. Furthermore, according to the fractal Einstein relation (Given and Mandelbrot 1983) ($d_w = \zeta + d_f$‡; $d_f$ is the fractal dimension of the Sierpinski carpet) and $d_w$ is given as follows:

$$d_w = \ln\left[\left(\frac{b-l}{b} + \frac{l}{b-l}\right)(b^2 - l^2)\right](\ln b)^{-1}. \tag{3}$$

As mentioned above, we must set $d_w = 2$ in order to get the HE lattice. As $b \to \infty$ with $d_f$ fixed, $d_w$ tends to 2. Therefore in this limit our Sierpinski carpets become the HE lattices of fractal dimension $d_f$.

† Gefen *et al* (1983) mentioned that this fixed point exists only when $b \to \infty$. Actually, however, the recursion relation always has the fixed point at $a = 2$.

‡ For the Sierpinski carpet, $d_f = \ln(b^2 - l^2)/\ln b$.

Next, consideration of the connectivity $Q$ (Mandelbrot 1982) enables us to find the condition for HE lattices. The connectivity represents the minimum fractal dimension for the horizontal sections. For Sierpinski carpets, $Q$ is equal to $\ln(b-l)/\ln b$. When $b\to\infty$, $Q$ reaches $d_f-1$. On the other hand, because

$$\int_0^L r^{d_f-1}\,\mathrm{d}r \sim L^{d_f} \tag{4}$$

when the minimum fractal dimension of horizontal sections is $d_f-1$, $d_f$ of all horizontal sections must be $d_f-1$. Otherwise the fractal dimension of the carpets themselves exceeds $d_f$. We conclude, therefore, that when the Sierpinski carpets become the HE lattices their horizontal sections have a unique fractal dimension, $d_f-1$. Our HE lattices contain the hypercubics of Gefen *et al* (1983). Indeed the translational invariance requires that the connectivity is $d_f-1$, but, even if $Q$ is equal to $d_f-1$, the carpets do not always have translational invariance. This is because there are several kinds of sections having a fractal dimension $d_f-1$.

In conclusion, we have constructed the HE lattices using the Sierpinski carpets. In the approximation of bond moving renormalisation for the dynamic property, the condition for the HE lattices is found to be that the connectivity $Q$ is equal to $d_f-1$. This condition does not require translational invariance. It is unclear whether this condition is sufficient for the thermodynamical property or not. Much more research is needed.

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